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Research Report CCS 556

OPTIMIZED CERTAINTY EQUIVALENTS
FOR DECISIONS UNDER UNCERTAINTY

By

A. Ben-Tal*
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**CENTER FOR
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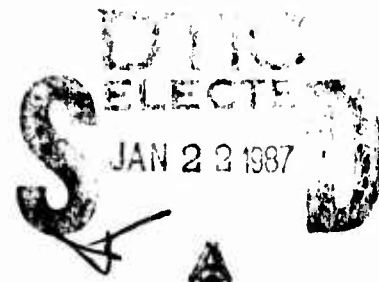
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ABSTRACT

A new approach is proposed for some models of decision-making under uncertainty, using optimized certainty-equivalents induced by expected-utility. Applications to production, investment and inventory models demonstrate the advantages of the new approach.

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KEY WORDS:

Expected Utility
Certainty Equivalent
Decision Making Under Uncertainty
Risk Aversion



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1. INTRODUCTION

Expected utility theory is "the major paradigm in decision making... . It has been used prescriptively in management science (especially decision analysis), predictively in finance and economics, descriptively by psychologists ... The **expected utility** (EU) model has consequently been the focus of much theoretical and empirical research ... [15].

In spite of its power, elegance and success, considerable criticism has been directed at the EU theory, specially when used **descriptively** to model the decision processes underlying risky choice. In this context, evidence refuting the validity of the EU model axioms is relevant, e.g. [1], [11].

A second usage, dominant in economics and finance, "is to view the EU model as **predictive** What matters is whether the model offers higher predictive accuracy than competing models of similar complexity What counts is whether the theory ... predicts behavior not used in the construction of the model", [15]. In this sense the EU model has been, to a large extent, successful.

To be specific, we consider two models of economic behavior under uncertainty: A **competitive firm under price uncertainty** [14],[9] and **investment in safe and in risky assets**, [2],[6],[7].

For the competitive firm, the EU model yields the fundamental result, that optimal production under uncertainty is less than that under (comparable) certainty. It also gives a sensible condition (necessary and sufficient) for production to start, [14].

In the investment model, diversification is prescribed by the EU model under a natural condition. An interesting example of the predictive power of the EU model is the following result due to Tobin, [17]:

"If a is the demand for risky investment when the return is a random variable X , then $a/(1+h)$ is the demand when the return is the random variable $(1+h)X$ ".

The above conclusions hold for any **risk-averse decision maker**, i.e. one for whom a random variable X is less desirable than a sure reward of EX . A fortunate aspect of the EU model is that risk-aversion is equivalent to the concavity of the utility function. Thus a fundamental attitude towards risk is characterized by a simple mathematical property.

Still, as a predictive tool the EU model is not without faults. It produces unnatural results, and it claims optimality for implausible behavior, even in the above production and investment models.

For a competitive firm under price uncertainty, [14], one would expect that an increase in the selling price will increase production. However, the EU model claims the opposite for certain risk-averse utilities. The dependence ¹ of

the optimal output on the fixed cost is another source of difficulty.

In the investment model, when the rate of return of the safe asset increases, one would expect part of the investment capital to switch from the risky asset to the safe asset. However, the EU model does not preclude the opposite behavior for certain utilities². Also it was established empirically³ that the elasticity of demand for cash balance is ≥ 1 ⁴, but here again the EU model leaves open the possibility of elasticity < 1 for certain risk-averse utility functions.

To avoid these pathologies (of the EU model), additional hypotheses are customarily imposed on the utility function U . These hypotheses are stated in terms of the **(absolute) Arrow-Pratt risk-aversion index**

$$r(z) = - \frac{U''(z)}{U'(z)} \quad (1.1)$$

and the **(relative) Arrow-Pratt risk-aversion index**,

$$R(z) = z r(z) \quad (1.2)$$

In the investment model, a typical postulate is [2]

" $r(z)$ is decreasing and $R(z)$ is increasing"

Similar conditions are imposed on $r(\cdot)$ and $R(\cdot)$ in the production model, [11]. These monotonicity conditions on $r(\cdot)$ and $R(\cdot)$ are conditions on the first three derivatives of U . Additional conditions are placed on the magnitude of R . Some of these hypotheses are controversial in themselves⁵. Taken together they severely restrict the applicability of the EU model.

The problem (central to decision making under uncertainty), of selecting a "most desirable" one from a set of random variables (RV's), supposes an **order** on the space of RV's, allowing in particular the comparison between RV's and constants (degenerate RV's). In particular, if a decision maker is indifferent between a RV Z and a constant z , then for him z is a **certainty equivalent**

¹ Called "paradoxical" in [9], and "seemingly paradoxical" in [14].

² To quote from [7], "such optimal behavior appears to be unlikely".

³ See references in [2], p. 103.

⁴ "Thus, the notion that security, in the particular form of cash balances, has a wealth elasticity of at least one, seems to be the only remaining explanation of the historical course of money holdings", [2], p. 104.

⁵ To quote from [2], p. 97: "The hypothesis of increasing relative risk aversion is not so easily confrontable with intuitive evidence".

(CE) of Z .

In the EU model, based on the classical theory of von Neumann and Morgenstern, the decision criterion is **maximal expected utility**. For decision making purposes, the EU criterion coincides with the **classical certainty equivalent** $C(\cdot)$ defined, for a RV Z and a given utility U , by

$$U(C(Z)) = EU(Z)$$

or

$$C(Z) = U^{-1}EU(Z) \quad (1.3)$$

Indeed, by the monotonicity of U , $EU(\cdot)$ and $C(\cdot)$ induce the same order on RV's,

$$EU(X) > EU(Y) \iff C(X) > C(Y)$$

Other reasonable definitions for CE's, based on expected utility, are possible. For example, the CE \hat{C} defined by

$$EU(Z - \hat{C}) = 0 \quad (1.4)$$

In this paper we propose the use, as a decision criterion, the CE

$$S_U(Z) = \sup_z \{z + EU(Z - z)\} \quad (1.5)$$

of the RV Z , with respect to the utility U . We call $S_U(Z)$ an **optimized certainty equivalent** (OCE) of Z . It was introduced, in a different context, in [4], [5]. Note that, unlike $C(\cdot)$, the CE's \hat{C} and S_U do not induce the same order as EU .

The advantages of the OCE approach, for predictive purposes, are demonstrated here by reexamining the above classical models of production and investment. In particular, the OCE approach (i) retains the successful predictions of the EU model (as listed above), (ii) does not require restrictive (third-derivative) conditions on U (thus the conclusions are valid for the whole class of risk-averse utilities), and (iii) is mathematically tractable, comparable in simplicity and elegance to the EU model ⁶.

For problems where the objective is **cost minimization**, a natural OCE

⁶ Not so for the CE defined by (1.4).

for a risk-avertter is

$$B_P(Z) = \inf_z \{z + EP(Z - z)\} \quad (1.6)$$

where $P(\cdot)$ is an increasing convex penalty function. This is not a departure from the classical theory, indeed, the von-Neumann Morgenstern principle of maximal expected utility could be alternatively developed as **minimal expected penalty**. Risk-aversion in this context means that a decision maker, who has to **pay** a random amount Z , prefers to pay the sure amount EZ . This is equivalent to acting on the basis of a convex penalty function P . We illustrate the OCE $B_P(\cdot)$, by applying it to a classical inventory model, [10].

A central property of the OCE is **shift additivity**,

$$S_U(Z + c) = S_U(Z) + c \quad (1.7)$$

for all utilities U , RV's Z and constants c ⁷. In contrast, the classical CE $C(\cdot)$ is shift additive only for **linear** or **exponential** U , [3]. For this reason, certain results (discussed in [3]), which in the EU model hold only for the exponential utility, hold in the OCE model for arbitrary utilities. Examples are the bridging of the gap between the buying and selling values of information, and the well known separation theorem in portfolio selection⁸.

The motivation for the OCE's S_U and B_P is provided in §2, together with basic properties. Associated functionals, useful in applications, are studied in §3. Section 4 is devoted to production under price uncertainty. The next two sections deal with investment in safe and in risky assets: The Arrow model [2] in §5, and a slight generalization in §6. We conclude with an application of the OCE B_P to an inventory model, in §7.

2. OPTIMIZED CERTAINTY EQUIVALENTS.

Consider a transaction where the ownership of a future value of a random variable Z is about to change hands. A price z , acceptable to a buyer, must satisfy (in some sense) the stochastic constraint

⁷ Shift additivity also holds for the CE given by (1.4), as well as for the CE in Yaari's new axiomatic system [19].

⁸ The proofs in [3] use only shift additivity.

$$z \leq Z \quad (2.1)$$

From the seller's point of view, the **minimal selling price** of Z is therefore the value of the stochastic program

$$\sup \{ z : z \leq Z \} \quad (2.2)$$

where the meaning of the stochastic constraint (2.1) is left vague on purpose.

Similarly, a price z acceptable to a seller must satisfy the stochastic constraint

$$z \geq Z \quad (2.3)$$

and the **maximal buying price** is the value of the stochastic program

$$\inf \{ z : z \geq Z \} \quad (2.4)$$

If Z is a degenerate RV, i.e. Z assumes a known value μ with probability 1, then (2.2) and (2.4) are dual linear programs, having μ as their common value.

One way to enforce the constraint (2.1) is to penalize its violation. Following [4] we replace (2.2) by the (unconstrained) problem

$$\sup_z \{ z + EU(Z - z) \} \quad (2.5)$$

where $U(\cdot)$ is a **penalty function**. In particular, if U is a monotone increasing function with $U(0) = 0$, then the term $EU(Z - z)$ in (2.5) penalizes [rewards] values z which violate [satisfy] (2.1) in the mean.

Specifically, we can regard $U(\cdot)$ as a **risk-averse utility function**, thus adding concavity to the above properties. Then (2.5) represents a two-stage approach to the stochastic program (2.2), with payments z and the observed future value of $Z - z$, discounted by $U(\cdot)$, [5].

Throughout this paper let \mathcal{U} be the class of **normalized utility functions**,⁹

⁹ The appropriate normalization for concave utilities, non-differentiable at 0, is $U(0) = 0, \lim_{z \rightarrow 0^+} \frac{U(z)}{z} = 1$

$$U = \left\{ \begin{array}{l} \text{strictly concave, differentiable, increasing} \\ \text{functions } U \text{ with } U(0) = 0, U'(0) = 1 \end{array} \right\} \quad (2.6)$$

Given $U \in U$, we define the **sellers optimized certainty equivalent** (SCE) of Z by (1.5)

$$S_U(Z) = \sup_z \{ z + EU(Z - z) \}$$

$S_U(Z)$, called the "new certainty equivalent" in [4], thus represents the minimal selling price of Z .

In the constrained stochastic program (2.4) we similarly penalize the violations of the constraint (2.3) by using a penalty function $P(\cdot)$, selected from the class of **normalized penalty functions**

$$P = \left\{ \begin{array}{l} \text{strictly convex, differentiable, increasing} \\ \text{functions } P \text{ with } P(0) = 0, P'(0) = 1 \end{array} \right\} \quad (2.7)$$

For any $P \in P$, we rewrite (2.4) as

$$\inf \{ z + EP(Z - z) \} \quad (2.8)$$

and define the **buyers optimized certainty equivalent** (BCE) of Z by (1.6)

$$B_P(Z) = \inf_z \{ z + EP(Z - z) \}$$

$B_P(Z)$ represents the maximal buying price of Z .

The transaction involving Z is made possible by the inequality

$$S_U(Z) \leq B_P(Z) \quad (2.9)$$

for all $U \in U$, $P \in P$, see Theorem 1(c) and (2.19). For such a pair $\{U, P\}$ the interval

$$[S_U(Z), B_P(Z)] \quad (2.10)$$

contains all prices acceptable to both seller and buyer, hence all relevant CE's.

THEOREM 1 (Properties of SCE).

(a) **Consistency.** For any $U \in U$ and a constant c ¹⁰

¹⁰ Considered as a degenerate RV.

$$S_U(c) = c \quad (2.11)$$

(b) **Shift additivity.** For any RV Z , constant c and $U \in \mathbf{U}$,

$$S_U(Z + c) = S_U(Z) + c \quad (1.7)$$

(c) **Risk aversion.** If U is an increasing, normalized utility function, then

$$S_U(Z) \leq EZ \quad \text{for all RV's } Z \quad (2.12)$$

if and only if U is concave. Equality holds in (2.12) if and only if U is linear.

(d) **Lower bound.** If Z is a RV, and $Z \geq z_{\min}$ with probability 1, then for any $U \in \mathbf{U}$,

$$z_{\min} \leq S_U(Z) \quad (2.13)$$

(e) **Stochastic dominance.** Let X, Y be RV's with compact supports. Then

$$S_U(X) \geq S_U(Y) \quad \text{for all } U \in \mathbf{U} \quad (2.14)$$

if and only if

$$EU(X) \geq EU(Y) \quad \text{for all } U \in \mathbf{U} \quad (2.15)$$

(f) **Concavity.** For any $U \in \mathbf{U}$, $0 < \alpha < 1$ and RV's X_0, X_1

$$S_U(\alpha X_1 + (1-\alpha)X_0) \geq \alpha S_U(X_1) + (1-\alpha)S_U(X_0) \quad (2.16)$$

(g) **Exhaustion.** Let Z be a nondegenerate RV with support $[z_{\min}, z_{\max}]$. Then for any $z_{\min} < z < EZ$, there is $U \in \mathbf{U}$ such that $S_U(Z) = z$.

PROOF. (a) For $U \in \mathbf{U}$, the gradient inequality implies

$$U(z) \leq U(0) + zU'(0) = z \quad \text{for all } z$$

and therefore

$$S_U(c) = \sup_z \{z + U(c-z)\} \leq \sup_z \{z + (c-z)\} = c.$$

$$\text{Also, } S_U(c) \geq \{c + U(c-c)\} = c.$$

$$(b) S_U(Z + c) = \sup_z \{z + EU(Z+c-z)\}$$

$$= c + \sup_z \{(z-c) + EU(Z - (z-c))\} = c + S_U(Z).$$

(c) By Jensen's inequality, U is concave if and only if

$$EU(Z) \leq U(EZ) \quad \text{for all RV's } Z$$

(with equality if and only if U is linear), which is equivalent to

$z + EU(Z-z) \leq z + U(EZ-z)$ for all z , and RV Z
and the result follows by taking suprema

$$\begin{aligned} S_U(Z) &= \sup_z \{z + EU(Z-z)\} \leq \sup_z \{z + U(EZ-z)\} \\ &= S_U(EZ) = EZ \quad \text{by (a).} \end{aligned}$$

(d) Since U is monotone,

$$\begin{aligned} S_U(Z) &\geq \sup_z \{z + U(z_{\min} - z)\} \\ &= z_{\min} \quad \text{by (a).} \end{aligned}$$

(e) $(2.15) \Rightarrow (2.14)$. Since each $U \in \mathbf{U}$ is increasing, (2.15) implies

$$z + EU(X-z) \geq z + EU(Y-z) \quad \text{for all } z, \text{ and all } U \in \mathbf{U}$$

and (2.14) follows by taking suprema.

$(2.14) \Rightarrow (2.15)$. Let z_X, z_Y be points where the suprema defining $S_U(X)$ and $S_U(Y)$ are attained, see Lemma 1. Then, for any $U \in \mathbf{U}$,

$$\begin{aligned} S_U(X) &= z_X + EU(X - z_X) \geq z_Y + EU(Y - z_Y), \quad \text{by (2.14)} \\ &\geq z_X + EU(Y - z_X) \end{aligned}$$

Therefore

$$EU(X - z_X) \geq EU(Y - z_X) \quad \text{for all } U \in \mathbf{U}, \text{ implying (2.15).}$$

(f) Let $0 < \alpha < 1$, and $X_\alpha = \alpha X_1 + (1 - \alpha)X_0$. Then by the concavity of U , for all z_0, z_1 ,

$$EU(X_\alpha - \alpha z_1 - (1 - \alpha)z_0) \geq \alpha EU(X_1 - z_1) + (1 - \alpha)EU(X_0 - z_0)$$

Adding $\alpha z_1 + (1 - \alpha)EU(X_0 - z_0)$ to both sides, and supremizing jointly with respect to z_1, z_0 , we get

$$\begin{aligned} S_U(X_\alpha) &\geq \sup_{z_1, z_0} \{\alpha[z_1 + EU(X_1 - z_1)] + (1 - \alpha)[z_0 + EU(X_0 - z_0)]\} \\ &= \alpha S_U(X_1) + (1 - \alpha)S_U(X_0) \end{aligned}$$

(g) We define for any $U \in \mathbf{U}$ and $\alpha > 0$ the function U_α by

$$U_\alpha(z) = \frac{1}{\alpha} U(\alpha z), \quad \forall z \quad (2.17)$$

Then $U_\alpha \in U$, and the corresponding SCE is ¹¹

$$\begin{aligned} S_{U_\alpha}(Z) &= \sup_z \left\{ z + \frac{1}{\alpha} U(\alpha(Z - z)) \right\} \\ &= \frac{1}{\alpha} \sup_z \{ z + U(\alpha Z - z) \} \\ &= \frac{1}{\alpha} S_U(\alpha Z) \end{aligned} \quad (2.18)$$

Since $S_{U_\alpha}(Z)$ is continuous in α , it follows from Lemma 2 (Appendix B) that for any

$$z_{\min} < z < EZ$$

there is $\alpha > 0$ such that $S_{U_\alpha}(Z) = z$, proving the exhaustive property of the SCE. \square

For the buyers OCE, an analogous theorem can be proved. In particular, the analogs of parts (c),(d) are

$$EZ \leq B_P(Z) \leq z_{\max} \quad (2.19)$$

for any RV Z , an upper bound z_{\max} and $P \in \mathbf{P}$. The concavity in (f) is replaced by convexity.

Theorem 1 lists properties which seem reasonable for any certainty equivalent. Properties (a) and (d) are natural and require no justification. The remaining properties will now be discussed one by one.

(b) To explain shift additivity consider a decision-maker indifferent between a lottery Z and a sure amount S . If 1 Dollar is added to all the possible outcomes of the lottery, then an addition of 1 Dollar to S will keep the decision maker indifferent.

(c) A decision maker is risk-averse (prefers EZ to Z) in our theory if and only if he is risk-averse in the EU theory, both risk-aversions equivalent to the

¹¹ In the two-stage model of the SCE (2.18), S_{U_α} can be interpreted as the present value of the sum of the payments z (at present) and $Z - z$ (in the future). Here α is the time discount factor.

concavity of the underlying utility U .

(e) In general,

$$EU(X) \geq EU(Y) \quad (2.20)$$

does not imply

$$S_U(X) \geq S_U(Y) \quad (2.21)$$

i.e. for a given U , (2.20) and (2.21) may induce different orders on the R.V.'s, see e.g. [5]. However, for the stronger preference of (2nd order) stochastic dominance, with the inequality holding for all $U \in \mathcal{U}$ ¹², (2.14) and (2.15) are equivalent.

(f) The concavity of $S_U(\cdot)$, for all $U \in \mathcal{U}$, expresses risk-aversion as aversion to variability. To gain insight consider the case of two independent RV's with the same mean and variance. The mixed RV $X_\alpha = \alpha X_1 + (1-\alpha)X_0$ has the same mean, but a smaller variance. Concavity of S_U means that the more centered RV X_α is preferred.

The risk-aversion inequality (2.12) is implied by (f): Let Z, Z_1, Z_2, \dots be independent, identically distributed RV's. Then by (f),

$$\begin{aligned} S_U\left(\sum_{i=1}^n Z_i/n\right) &\geq \frac{1}{n} \sum_{i=1}^n S_U(Z_i) \\ &= S_U(Z) \end{aligned}$$

As $n \rightarrow \infty$, (2.12) follows by the strong law of large numbers.

In contrast, the classical CE $U^{-1}EU(\cdot)$ is not necessarily concave for all concave U .

(g) This property simply means that any value between z_{\min} and EZ is the certainty equivalent of some risk-averse decision maker.

The question of the attainment of the supremum in (1.5) is settled in the following:

LEMMA 1. Let the RV Z have support $[z_{\min}, z_{\max}]$, with finite z_{\min} and z_{\max} . Then the supremum in (1.5) is attained at some z_g ,

¹² In which case Y is called riskier than X .

$$z_{\min} \leq z_S \leq z_{\max}, \quad (2.22)$$

which is the unique solution of

$$EU'(Z - z_S) = 1, \quad (2.23)$$

so that

$$S_U(Z) = z_S + EU(Z - z_S) \quad (2.24)$$

PROOF. The function

$$u(z) = z + EU(Z - z) \quad (2.25)$$

is concave, so its supremum is attained at a stationary point z_S satisfying (2.23) (obtained by differentiating (2.25)). The uniqueness of z_S follows from the fact that U' is strictly decreasing, implied by the strict concavity of U . From (2.23) it also follows that

$$U'(z_{\min} - z_S) \geq 1$$

and

$$U'(z_{\max} - z_S) \leq 1$$

proving (2.22). □

Similarly, the infimum in (1.6) is attained at some z_B in $[z_{\min}, z_{\max}]$, which is the unique solution of

$$EP'(Z - z_B) = 1 \quad (2.26)$$

so that

$$P_P(Z) = z_B + EP(Z - z_B) \quad (2.27)$$

In the above discussion of S_U and B_P , the utility U and the penalty P are unrelated. In the following two examples we illustrate our results using pairs $\{P, U\}$ related by ¹³

$$P(z) = -U(-z), \quad \forall z \quad (2.28)$$

EXAMPLE 1. (Exponential utility). Here

¹³ Indeed (2.28) is a 1:1 correspondence between U and P . It also brings some symmetry into the discussion, since U and P (of (2.28)) penalize deviations in the same way. Another natural correspondence between U and P is $P = U^{-1}$.

$$U(z) = 1 - e^{-z}, \quad \forall z \quad (2.29)$$

and equation (2.23) becomes $Ee^{-Z+z} = 1$, giving $z_S = -\log Ee^{-Z}$ and the same value for the certainty equivalent

$$S_U(Z) = -\log Ee^{-Z} \quad (2.30)$$

A special feature of the exponential utility function (2.29) is that the classical CE (1.3) becomes

$$U^{-1}EU(Z) = -\log Ee^{-Z}$$

showing that for the exponential utility, the certainty equivalents (1.5) and (1.3) coincide.

The corresponding penalty, by (2.28), is

$$P(z) = -1 + e^z \quad (2.31)$$

giving the BCE

$$B_P(Z) = \log Ee^Z \quad (2.32)$$

EXAMPLE 2. (Quadratic utility).¹⁴ Here

$$U(z) = z - \frac{1}{2} z^2 \quad z \leq 1 \quad (2.33)$$

and for a RV Z with $z_{\max} \leq 1$, $EZ = \mu$ and variance σ^2 , equation (2.23) gives $z_S = \mu$, and by (2.24)

$$S_U(Z) = \mu - \frac{1}{2} \sigma^2 \quad (2.34)$$

The corresponding penalty (2.28) is

$$P(z) = z + \frac{1}{2} z^2 \quad (2.35)$$

with BCE

$$B_P(Z) = \mu + \frac{1}{2} \sigma^2 \quad (2.36)$$

Note that here $B_P(Z) - S_U(Z) = \sigma^2$. We show in (3.12) that for small σ^2 , $B_P(Z) - S_U(Z)$ is approximately linear in σ^2 for all $U \in \mathcal{U}$ and $P \in \mathcal{P}$.

COROLLARY 1. In both the exponential and quadratic utilities

¹⁴ The restriction $z \leq 1$ in (2.33) guarantees that U is increasing throughout its domain.

$$S_U\left(\sum_{i=1}^n Z_i\right) = \sum_{i=1}^n S_U(Z_i) \quad (2.37)$$

for independent RV's $\{Z_1, Z_2, \dots, Z_n\}$ ¹⁵. □

EXAMPLE 3. For the so-called **hybrid model** ([3], [8], [16]), with exponential utility U and a normally distributed RV $Z \sim N(\mu, \sigma^2)$,

$$S_U(Z) = \mu - \frac{1}{2}\sigma^2$$

3. FUNCTIONALS AND APPROXIMATIONS.

Let $Z = (Z_i)$ be a RV in R^n , with expectation μ (vector) and covariance matrix Σ (if $n = 1$ then as above $\Sigma = \sigma^2$). For any vector $y \in R^n$, the **inner product**

$$y \cdot Z = \sum_{i=1}^n y_i Z_i$$

is a scalar RV. Given $U \in \mathbf{U}$ and $P \in \mathbf{P}$, the corresponding CE's of $y \cdot Z$ are taken as functionals in y , the **SCE functional**

$$s(y) = S_U(y \cdot Z), \quad (3.1)$$

and the **BCE functional**

$$b(y) = B_P(y \cdot Z). \quad (3.2)$$

We collect properties of the SCE functional in the following theorem, whose proof appears in Appendix A. The analogous statements of the BCE functional are omitted.

THEOREM 2. Let $U \in \mathbf{U}$ be twice continuously differentiable, and let Z and $s(\cdot)$ be as above. Then:

(a) The functional s is concave, and given by

$$s(y) = z_S(y) + EU(y \cdot Z - z_S(y)) \quad (3.3)$$

where $z_S(y)$ is the unique solution z of

¹⁵ The classical CE (1.3) is additive, for independent RV's, if U is exponential but not if U is quadratic.

$$EU'(y \cdot Z - z) = 1 \quad (3.4)$$

(b) Moreover,

$$s(0) = 0, \quad \nabla s(0) = \mu, \quad \nabla^2 s(0) = U''(0)\Sigma \quad (3.5)$$

$$z_s(0) = 0, \quad \nabla z_s(0) = \mu \quad (3.6)$$

and if U is three times continuously differentiable,

$$\nabla^2 z_s(0) = \frac{U'''(0)}{U''(0)} \quad (3.7)$$

□

Theorem 2, and the corresponding statements about the functional $b(\cdot)$, can be used to obtain the following approximations of the functionals $s(\cdot)$, $b(\cdot)$ based on their Taylor expansions around $y = 0$.

COROLLARY 2. If U , P are three times continuously differentiable then

$$s(y) = \mu \cdot y + \frac{1}{2} U''(0) y \cdot \Sigma y + o(\|y\|^2) \quad (3.8)$$

$$b(y) = \mu \cdot y + \frac{1}{2} P''(0) y \cdot \Sigma y + o(\|y\|^2) \quad (3.9)$$

□

REMARKS

(a) In particular, for $n = 1$ and $y = 1$, it follows from (3.8) and (3.9) that

$$S_U(Z) = \mu + \frac{1}{2} U''(0) \sigma^2 + o(\sigma^2) \quad (3.10)$$

$$= \mu - \frac{1}{2} r(0) \sigma^2 + o(\sigma^2)$$

where $r(\cdot)$ is the Arrow-Pratt risk-aversion index (1.1). Similarly,

$$B_P(Z) = \mu + \frac{1}{2} P''(0) \sigma^2 + o(\sigma^2) \quad (3.11)$$

and therefore

$$B_P(Z) - S_U(Z) = \frac{1}{2} (P''(0) - U''(0)) \sigma^2 + o(\sigma^2) \quad (3.12)$$

(b) We also note that the approximation (3.12) is exact if

- (i) U is quadratic, or
- (ii) U is exponential, Z is normal.

(c) By differentiating, and calculating the Taylor expansion of the classical CE of $y \cdot Z$

$$ce(y) = U^{-1}EU(y \cdot Z) \quad (3.13)$$

it follows that $ce(y)$ is approximated by the right-hand side of (3.12). Thus we have the unexpected result

$$ce(y) - s(y) = o(\|y\|^2) \quad (3.14)$$

showing that for small y the CE functionals (3.1) and (3.13) are practically the same.

4. COMPETITIVE FIRM UNDER UNCERTAINTY.

The first application of the OCE is to the classical model studied by Sandmo [14], see also [9, § 5.2]. A firm sells its **output** q at **price** P , which is a RV with a known distribution function and expected value $EP = \mu$. Let $C(q)$ be the **total cost** of producing q , which consists of a **fixed cost** B and a **variable cost** $c(q)$,

$$C(q) = c(q) + B$$

The function $c(\cdot)$ is assumed normalized, increasing and strictly convex,

$$c(0) = 0, \quad c'(q) > 0, \quad c''(q) > 0 \quad \forall q \geq 0 \quad (4.1)$$

The firm has a strictly concave utility function U , i.e.

$$U' > 0, \quad U'' < 0$$

which is normalized so that $U(0) = 0$, $U'(0) = 1$. The objective is to maximize **profit**

$$\pi(q) = qP - c(q) - B$$

which is a RV. The classical CE (1.3) is used in Sandmo's analysis, so that the model studied is

$$\max_{q \geq 0} U^{-1}EU(\pi(q))$$

or equivalently,

$$\max_{q \geq 0} EU(\pi(q)) \quad (4.2)$$

Here we analyze the same model using the SCE (1.5). The objective of the firm is therefore

$$\max_{q \geq 0} S_U(\pi(q)) \quad (4.3)$$

Now

$$\begin{aligned} \max_{q \geq 0} S_U(\pi(q)) &= \max_{q \geq 0} S_U(qP - c(q) - B) \\ &= \max_{q \geq 0} \{S_U(qP) - c(q)\} - B \end{aligned}$$

by (1.7). We conclude:

PROPOSITION 1. The optimal production output q^* is independent of the fixed cost B . □

This result is in sharp contrast to the expected utility model (4.2) where the optimal output \bar{q} depends on the fixed cost B : \bar{q} increases [decreases] with B if the Arrow-Pratt index $r(\cdot)$ is an increasing [decreasing] function; the dependence is ambiguous for utilities for which $r(\cdot)$ is not monotone.

Note that the objective function in (4.3) is

$$f(q) = s(q) - c(q) \quad (4.4)$$

where $s(\cdot)$ is the SCE functional (3.1). The function f is concave by Theorem 2 and the assumptions on c . Therefore, the optimal solution q^* of (4.3) is positive if and only if $f'(0) > 0$. By (3.5) $s'(0) = \mu$, so

$$q^* > 0 \text{ if and only if } \mu > c'(0) \quad (4.5)$$

in agreement with the expected utility model (4.2). We assume from now on that

$$\mu > c'(0)$$

A central result in the theory of production under uncertainty is that, for the risk-averse firm (i.e. concave utility function), the optimal production under uncertainty is less than the corresponding optimal production q_{cer} under certainty, that is for P a degenerate RV with value μ . We will prove now that the same result holds for the model (4.3). First recall that the optimality condition

for q_{cer} is that marginal cost equals marginal revenue

$$c'(q_{cer}) = \mu \quad (4.6)$$

PROPOSITION 2. $q^* < q_{cer}$ for all $U \in \mathbf{U}$.

PROOF. The optimality condition for q^* is

$$0 = f'(q^*) = s'(q^*) - c'(q^*) \quad (4.7)$$

By Theorem 2

$$s(q) = z(q) + EU(qP - z(q)) \quad (4.8)$$

where $z(q)$ is a differentiable function, uniquely determined by the equation

$$EU'(qP - z(q)) = 1 \quad (4.9)$$

Differentiating (4.8) with respect to q yields

$$\begin{aligned} s'(q) &= z'(q) + E\{(P - z'(q))U'(qP - z(q))\} \\ &= z'(q)[1 - EU'(qP - z(q))] + EPU'(qP - z(q)) \end{aligned}$$

Using (4.9) we then get

$$s'(q) = E\{PU'(qP - z(q))\} \quad (4.10)$$

Therefore the optimality condition (4.7) becomes

$$EPU'(q^*P - z(q^*)) = c'(q^*) \quad (4.11)$$

Multiplying (4.9) by μ and subtracting from (4.11) we get

$$E(P - \mu)U'(q^*P - z(q^*)) = c'(q^*) - \mu \quad (4.12)$$

or

$$E\{Zh(Z)\} = c'(q^*) - \mu \quad (4.13)$$

where we denote

$$Z := P - \mu, \quad h(Z) := U'(q^*Z + q^*\mu - z(q^*))$$

Since $U \in \mathbf{U}$, it follows that h is positive and decreasing, and it can then be shown (see e.g. [9, p. 249]) that

$$E\{Zh(Z)\} < h(0)E\{Z\}$$

but $E\{Z\} = E\{P - \mu\} = 0$, and so, by (4.13),

$$c'(q^*) < \mu$$

and by using (4.6)

$$c'(q^*) < c'(q_{cer})$$

and since c' is increasing,

$$q^* < q_{cer}$$

□

EFFECT OF PROFITS TAX

Suppose there is a proportional **profits tax** at rate $0 < t < 1$, so that the profit after tax is

$$\pi(q) = (1 - t)(qP - C(q))$$

As before, the firm seeks the optimal solution q^* of (4.3), which here becomes

$$\begin{aligned} \max_{q \geq 0} S_U(\pi(q)) &= \max_{q \geq 0} S_U((1 - t)(qP - c(q) - B)) \\ &= \max_{q \geq 0} \{S_U((1 - t)qP) - (1 - t)c(q)\} - (1 - t)B \end{aligned}$$

which can be rewritten, using the SCE functional $s(\cdot)$ and omitting the constant $(1 - t)B$,

$$\max_{q \geq 0} s((1 - t)q) - (1 - t)c(q)$$

Let the optimal solution be $\bar{q} = \bar{q}(t)$. The optimality condition here is

$$(1 - t)s'((1 - t)\bar{q}) - (1 - t)c'(\bar{q}) = 0$$

giving the identity (in t),

$$s'((1 - t)\bar{q}(t)) \equiv c'(\bar{q}(t))$$

which, after differentiating (with respect to t),

$$[(1 - t)\bar{q}'(t) - \bar{q}(t)]s''((1 - t)\bar{q}) = \bar{q}'(t)c''(\bar{q})$$

and rearranging terms, gives

$$\bar{q}'(t)\{c''(\bar{q}) - (1 - t)s''((1 - t)\bar{q})\} = -\bar{q}(t)s''((1 - t)\bar{q}) \quad (4.14)$$

The coefficient of $\bar{q}'(t)$ is positive since $c'' > 0$ and $s(\cdot)$ is concave (Theorem 2(a)). The right-hand side of (4.14) is positive since $\bar{q} > 0$, $s'' < 0$. Therefore, by (4.14),

$$\bar{q}'(t) > 0$$

and we proved:

PROPOSITION 3. An increase in profit tax causes the firm to increase production. \square

In the classical expected utility case the effect of taxation depends on third-derivative assumptions, and is undetermined if the relative risk-aversion index $R(\cdot)$ is not monotone.

EFFECT OF PRICE INCREASE.

If price were to increase from P to $P + \epsilon$ (ϵ fixed), then the corresponding optimal output $\bar{q}(\epsilon)$ is the solution of

$$\max_{q \geq 0} \{S_P((P + \epsilon)q) - c(q)\} = \max_{q \geq 0} \{s(q) + \epsilon q - c(q)\}$$

The optimality condition for $\bar{q}(\epsilon)$ is

$$s'(\bar{q}(\epsilon)) + \epsilon = c'(\bar{q}(\epsilon))$$

Differentiating with respect to ϵ we get

$$\bar{q}'(\epsilon) s''(\bar{q}(\epsilon)) + 1 = \bar{q}'(\epsilon) c''(\bar{q}(\epsilon))$$

hence

$$\bar{q}'(\epsilon) = \frac{1}{c''(\bar{q}) - s''(\bar{q})} > 0$$

by the convexity of c and the concavity of s . We have so proved:

PROPOSITION 4. An increase in selling price causes the firm to increase production. \square

This highly intuitive result is proved in the expected utility case only under the assumption that $r(\cdot)$ is non-increasing.

5. INVESTMENT IN ONE RISKY AND IN ONE SAFE ASSETS: THE ARROW MODEL.

Recall the classical model [2] of investment in a risky/safe pair of assets, concerning an individual with utility $U \in \mathcal{U}$ and initial wealth A . The **decision variable** is the amount a to be invested in the risky asset, so that $m = A - a$ is the amount invested in the safe asset (cash).

The rate of return in the risky asset is a RV X .

The final wealth of the individual is then

$$Y = A - a + (1 + X) a = A + a X$$

In [2] the model is analyzed via the maximal EU principle, so the optimal investment a^* is the solution of

$$\max_{0 \leq a \leq A} EU(A + aX) \quad (5.1)$$

or equivalently

$$\max_{0 \leq a \leq A} C(A + aX) \quad (5.2)$$

Some of the important results in [2] are:

(A1) $a^* > 0$ if and only if $EX > 0$.

(A2) a^* increases with wealth (i.e. $\frac{da^*}{dA} \geq 0$) if the absolute risk aversion index $r(\cdot)$ is decreasing.

(A3) The wealth elasticity of the demand for cash balance (investment in the safe asset)

$$\frac{Em}{EA} := \frac{dm/dA}{m/A} \quad \text{is at least one} \quad (5.2)$$

if the relative risk-aversion index

$$R(z) = -z \frac{U''(z)}{U'(z)} \quad \text{is increasing} \quad (5.3)$$

Arrow [2] postulated that reasonable utility functions **should** satisfy (5.3), since the empirical evidence for (5.2) is strong, see the references in [2], p. 103.

We analyze this investment problem using the SCE criterion, i.e.

$$\max_{0 \leq a \leq A} S_U(A + aX) \quad (5.4)$$

which, by (1.7) is equivalent to

$$\max_{0 \leq a \leq A} S_U(aX) + A$$

Let a^* be the optimal solution. Using the SCE functional $s(\cdot)$, a^* is in fact the solution of

$$\max_{0 \leq a \leq A} s(a) \quad (5.5)$$

Now, since $s(\cdot)$ is concave

$$a^* > 0 \quad \text{if and only if} \quad s'(0) > 0$$

but by (3.5) $s'(0) = EX$, and we recover the result (A1).

Assuming (as in [2]) an inner optimal solution (**diversification**)

$$0 < a^* < A \quad (5.6)$$

we conclude here, in contrast to (A2), that

$$\frac{da^*}{dA} = 0 \quad (5.7)$$

i.e. the optimal investment is independent of wealth.¹⁶

An immediate consequence of (5.7) is

$$\frac{Em}{EA} > 1 \quad \forall U \in \mathcal{U}$$

indeed

$$\frac{Em}{EA} = \frac{A}{m} \frac{dm}{dA} = \frac{A}{A - a^*} \frac{d(A - a^*)}{dA} = \frac{A}{A - a^*} \left(1 - \frac{da^*}{dA}\right) = \frac{A}{A - a^*} > 1$$

proving (5.2) for all risk-averse utilities. Thus, in the SCE model, there is no need for the controversial postulate (5.3).

The quadratic utility (2.33)

$$U(z) = z - \frac{1}{2}z^2 \quad z \leq 1$$

violates both of Arrow's postulates (r decreasing, R increasing), and is consequently "banned" from the EU model. In the SCE model, on the other hand, the

¹⁶ However, initial wealth will in general determine when diversification will be optimal, i.e. when (5.6) will hold.

quadratic utility is acceptable ¹⁷. For the quadratic utility the optimal investment a^* is the optimal solution of

$$0 \leq a \leq A \quad \{s(a) = \mu a - \frac{1}{2}\sigma^2 a^2\}$$

where $\mu = EX$, $\sigma^2 = \text{Var}(X)$. Therefore

$$a^* = \begin{cases} \mu/\sigma^2 & \text{if } 0 < \mu/\sigma^2 < A \\ A & \text{if } \mu/\sigma^2 \geq A \end{cases}$$

showing that, for the full range of A values, $a^*(A)$ is nondecreasing, in agreement with (A2). Moreover, if diversification is optimal, then

$$\frac{Em}{EA} = \frac{A}{A - \mu/\sigma^2} > 1$$

Following [2] we consider the effects on optimal investment, of shifts in the RV X . Let h be the **shift parameter**, and assume that the **shifted** RV $X(h)$ is a differentiable function of h , with $X(0) = X$. Examples are

$$\begin{aligned} X(h) &= X + h \quad (\text{additive shift}), \\ X(h) &= (1 + h)X \quad (\text{multiplicative shift}). \end{aligned}$$

For the shifted problem, the objective is

$$0 \leq a \leq A \quad \max_{a \leq A} S_U(aX(h)) \quad (5.8)$$

Let $a(h)$ be the optimal solution of (5.8), in particular $a(0) = a^*$. Now

$$S_U(aX(h)) = \xi(a) + EU(aX(h) - \xi(a)) \quad (5.9)$$

where $\xi(a)$ is the unique solution of

$$EU'(aX(h) - \xi(a)) = 1 \quad (5.10)$$

The optimality condition for $a(h)$ is

$$\frac{d}{da} \{ \xi(a) + EU(aX(h) - \xi(a)) \}$$

which gives (using (5.10)) the following identities in h

$$E\{X(h)U'(a(h)X(h) - \xi(a(h)))\} \equiv 0 \quad (5.11)$$

¹⁷ Assuming $0 \leq X \leq 1$

$$E\{U'(a(h)X(h) - \xi(a(h)) - \xi(a(h)))\} \equiv 1 \quad (5.12)$$

Differentiating (5.11) with respect to h we get, denoting $Z = aX(h) - \xi(a(h))$,

$$\dot{a}(h)E\{U''(Z)X(X - \xi'(a(h)))\} + E\{\dot{X}(h)[U'(Z) + \xi(h)X(h)U''(Z)]\} \quad (5.13)$$

where $\dot{a}(h) = \frac{d}{dh}a(h)$ and similarly for $\dot{X}(h)$.

The second order optimality condition for $a(h)$, $\frac{d^2}{da^2}S_U(aX(h)) \geq 0$, is here

$$EU''(Z)X(X - \xi'(a(h))) \geq 0$$

hence, by (5.13),

$$\text{sign of } \dot{a}(h) = \text{sign of } E\{\dot{X}(h)[U'(Z) + aXU''(Z)]\}$$

exactly the same condition for the sign of $\frac{d}{dh}a(h)$ as in [2], p. 105, eq. (18).

Therefore, the conclusions of the EU model are also valid for the SCE model. In particular:

PROPOSITION 5. As a function of the shift parameter h ,
 $a(h)$ increases for additive shift,
 $a(h)$ decreases for multiplicative shift. □

These results are illustrated for the quadratic utility. There

$$a^* = \frac{EX}{\text{Var}(X)}$$

and

$$a(h) = a^* + \frac{h}{\text{Var}(X)} \quad \text{for an additive shift}$$

$$a(h) = \frac{1}{1+h}a^* \quad \text{for a multiplicative shift} \quad (5.14)$$

In fact, (5.14) holds for arbitrary $U \in \mathcal{U}$, a result proved in [17] for the EU model.

PROPOSITION 6. If a^* is the demand for the risky asset when the return is the RV X , then $a(h) = a^*/(1+h)$ is the demand when the return is $(1+h)X$.

PROOF. The optimality condition for a^* is

$$E\{U'(a^*X - \xi^*)X\} = 0 \quad (5.15)$$

where ξ^* is the unique solution of

$$EU'(a^*X - \xi^*) = 1 \quad (5.16)$$

The optimality conditions for $a(h)$ are given by (5.11), (5.12). Now, for

$$a(h) = \frac{1}{1+h} a^*,$$

$$a(h)X(h) = a^*X \quad (5.17)$$

and it follows, by comparing (5.12) with (5.16), that

$$\xi(a(h)) = \xi^*$$

Substituting this in (5.11) and using (5.17), we see that (5.11) is equivalent to (5.16), and that $a(h) = a^*/1+h$ indeed satisfies the optimality conditions (5.11), (5.12). \square

6. INVESTMENT IN A RISKY/SAFE PAIR OF ASSETS: AN EXTENSION

We study the model discussed in [6] and [7], which is an extension of the model in §5. The analysis applies to a fixed time interval, say a year. An investor allocates a proportion $0 \leq k \leq 1$ of his investment capital W_0 to a risky asset, and proportion $1-k$ of W_0 to a safe asset where the total annual return per dollar invested is $\tau \geq 1$. The total annual return t per dollar invested in the risky asset, is a nonnegative RV. The investor's **total annual return** is

$$kW_0t + (1-k)W_0\tau$$

and for a utility function U , the optimal allocation k^* is the solution of

$$\max_{0 \leq k \leq 1} EU(kW_0t + (1-k)W_0\tau) \quad (6.1)$$

The model of §5, is a special case with $W_0 = A$, $t = 1+X$, $kW_0 = a$, $\tau = 1$.

It is assumed in [6], [7] that $U' > 0$ and $U'' < 0$, thus we assume without loss of generality that $U \in \mathcal{U}$.

One of the main issues in [7] is the effect of an increase in the safe asset return τ on the optimal allocation. The following are proved:

(F1) An investor maximizing expected utility will diversify (invest a positive amount in each of the assets) if and only if

$$\frac{EtU'(W_0t)}{EU'(W_0t)} < \tau < E(t) \quad (6.2)$$

(F2) Given (6.2) he will **increase** the proportion invested in the safe asset when τ increases if either

- (a) the absolute risk aversion index $r(\cdot)$ is nondecreasing, or
- (b) the relative risk aversion index $R(\cdot)$ is at most 1.

The same model is now analyzed using the SCE approach, i.e. with the objective

$$\max_{0 \leq k \leq 1} S_U(kW_0t + (1-k)W_0\tau)$$

Using (1.7) and the definition (3.1), the objective becomes

$$\max_{0 \leq k \leq 1} \{(1-k)W_0\tau + s(W_0k)\} \quad (6.3)$$

The following proposition, proved in Appendix C, gives the analogs of results (F1), (F2) in the SCE model.

PROPOSITION 7.

(a) The SCE maximizing investor will diversify if and only if

$$EtU'(W_0t - \eta) < \tau < E(t) \quad (6.4)$$

where η is the unique solution of

$$EU'(W_0t - \eta) = 1 \quad (6.5)$$

(b) Given (6.4), he will **increase** the proportion invested in the safe asset when τ **increases**. □

Comparing part (b) with (F2), we see that plausible behavior (k^* increases with τ) holds in the SCE model for all $U \in \mathcal{U}$, but in the EU model only for some U .

We illustrate Proposition 7 in the case of the quadratic utility (2.33). Here the optimal proportion invested in the risky asset is:

$$k^* = \begin{cases} 0 & \text{if } \tau > E(t) \\ \frac{E(t) - \tau}{W_0\sigma^2} & \text{if } E(t) - W_0\sigma^2 \leq \tau \leq E(t) \\ 1 & \text{if } E(t) - W_0\sigma^2 > \tau \end{cases} \quad (6.6)$$

where σ^2 is the variance of t . Thus k^* is increasing in $E(t)$, decreasing with σ^2 and decreasing with τ (so that, the proportion $1-k^*$ invested in the safe asset is

increasing with safe asset return r). These are reasonable reactions of a risk-averse investor.

We also see from (6.6) that k^* **decreases** when the investment capital W_0 **increases**. This result holds for arbitrary $U \in \mathbf{U}$, see the next proposition (proved in Appendix C). In the EU model, the effect of W_0 on k^* depends on the relative risk-aversion index, see [6].

PROPOSITION 8. If the investment capital increases, then the SCE-maximizing investor will increase the proportion invested in the safe asset. \square

Following the analysis in [2] and §5, we consider now the elasticity of cash-balance (with respect to W_0). Here the cash balance (the amount invested in the safe asset) is

$$m = (1 - k^*) W_0$$

and the elasticity in question is $\frac{Em}{EW_0}$.

PROPOSITION 9. For every SCE-maximizing investor with $U \in \mathbf{U}$,

$$\frac{Em}{EW_0} \geq 1$$

PROOF.

$$\frac{Em}{EW_0} = \frac{dm/dW_0}{m/W_0} = \frac{1 - k^*(W_0) - W_0 \frac{dk^*(W_0)}{dW_0}}{1 - k^*(W_0)}$$

hence

$$\frac{Em}{EW_0} \geq 1 \text{ if and only if } \frac{dk^*(W_0)}{dW_0} \leq 0 \quad (6.7)$$

and the proof is completed by Proposition 8. \square

The equivalence in (6.7) shows that the empirically observed fact that $Em/EW_0 \geq 1$ can be explained **only** by the result established in Proposition 8 that $dk^*/dW_0 \leq 0$, a result which is not necessarily true for many utilities in the EU analysis.

7. AN INVENTORY MODEL.

Consider a classical inventory problem, e.g. ([10], §2.5), where **demand** (for the item in question) occurs at a rate of d units per day. Orders are received immediately after they are placed with the supplier. Unsatisfied demand is backlogged until it can be satisfied. **Holding cost** is h per unit per day, **shortage cost** (penalty for unsatisfied demand) is p per unit per day. **Material cost** is c per unit, and there is a fixed **transaction cost** of k per order.

An (S, s) **policy** is used (whenever stock level falls below s , order up to S). The aim is to minimize **total cost** per period, given by

$$TC = \frac{hS^2}{2(S-s)} + \frac{ps^2}{2(S-s)} + \frac{kd}{S-s} + cd \quad (7.1)$$

and the optimal parameters of the policy are given by

$$s^* = \sqrt{\frac{2kd}{h}} \sqrt{\frac{p}{p+h}} \quad (7.2)$$

$$s^* = -\frac{h}{p}S^* \quad (7.3)$$

The least-cost order quantity $Q^* = S^* - s^*$ is then

$$Q^* = \sqrt{\frac{2kd}{h}} \sqrt{\frac{p+h}{p}} \quad (7.4)$$

We now analyze this model under the assumption that the demand D is a nonnegative RV. To compare our results with the deterministic case we assume that $E\{D\} = d$. We also denote the variance of D by σ^2 . Since we have a cost minimization problem, we use the BCE criterion, and so minimize the BCE of the total cost (7.1)

$$\min_{s, S} B_P\left(\frac{hS^2}{2(S-s)} + \frac{ps^2}{2(S-s)} + \frac{kD}{S-s} + cD\right)$$

which, by (1.7), reduces to

$$\min_{s, S} \{f(S, s) := \frac{hS^2}{2(S-s)} + \frac{ps^2}{2(S-s)} + b\left(\frac{k}{S-s} + c\right)\} \quad (7.5)$$

where $b(\cdot)$ is the BCE functional of (3.2).

The first order necessary conditions for an optimal pair (\bar{S}, \bar{s}) are

$$\frac{\partial f(\bar{S}, \bar{s})}{\partial S} = \quad (7.6)$$

$$= \frac{1}{2(\bar{S} - \bar{s})^2} \{2(\bar{S} - \bar{s})h\bar{S} - (h\bar{S}^2 + p\bar{s}^2) - 2kb'(\frac{k}{\bar{S} - \bar{s}} + c)\} = 0$$

$$\frac{\partial f(\bar{S}, \bar{s})}{\partial s} = \quad (7.7)$$

$$= \frac{1}{2(\bar{S} - \bar{s})^2} \{2(\bar{S} - \bar{s})p\bar{s} + (h\bar{S}^2 + p\bar{s}^2) + 2kb'(\frac{k}{\bar{S} - \bar{s}} + c)\} = 0$$

By adding (7.6) and (7.7) we obtain the relation

$$\bar{s} = -\frac{h}{p}\bar{S} \quad (7.8)$$

in analogy with (7.3). Substituting (7.8) in (7.6) we get an equation for \bar{S} :

$$T(\bar{S}) := \bar{S}^2 h(1 + \frac{h}{p}) - 2kb'(\frac{k}{\bar{S}(1 + \frac{h}{p})} + c) = 0 \quad (7.9)$$

Recall that $b(\cdot)$ is convex (proved analogously to the concavity of $s(\cdot)$ in Theorem 2(a)). Therefore the function $T(\cdot)$ in (7.9) is increasing for $\bar{S} \geq 0$. Moreover, $b'(y) > 0$ for all $y > 0$ since $b'(0) = E\{D\} = d > 0$. Therefore,

$$T(0) < 0 \text{ and } T(\infty) = \infty$$

and equation (7.9) has a unique solution \bar{S} .

Comparing the solution (\bar{S}, \bar{s}) to the deterministic solution (S^*, s^*) (given in (7.2),(7.3)), we get:

PROPOSITION 10. For every BCE-maximizing decision-maker (i.e. for every $P \in \mathbf{P}$) the optimal order quantity under uncertainty $\bar{Q} := \bar{S} - \bar{s}$ is larger than the optimal order quantity under certainty Q^* .

PROOF. By (7.9) with $\theta := 1 + \frac{h}{p}$,

$$\bar{S}^2 h \theta = 2kb'(\frac{k}{\bar{S}\theta} + c) > 2kb'(0) = 2kd$$

hence

$$\bar{S} > \sqrt{\frac{2kd}{h\theta}} = \sqrt{\frac{2kd}{h}} \sqrt{\frac{p+h}{p}} = S^*$$

Using (7.8) and (7.3) we get a comparison for the other parameters

$$\bar{s} = -\frac{h}{p}\bar{S} < -\frac{h}{p}S^* = s^*$$

and finally

$$\bar{Q} = \bar{S} - \bar{s} > S^* - s^* = Q^*$$

□

The effect of changes in the cost parameters on \bar{S} can be determined from the optimality equation (7.9). These results are summarized in the Table 1, and compared to the analogous results in the deterministic case.

Increase in:	Effect on \bar{S} (stochastic demand)	Effect on S^* (deterministic demand)
Shortage cost p	increase	increase
Holding cost h	decrease	decrease
Transaction cost k	increase	increase
Material cost c	increase	no effect

TABLE 1. Effects of cost changes on policy parameters.

Thus we have the same effects of cost changes in the deterministic and stochastic cases, except for the material cost c : In the deterministic case the optimal policy is independent of c (see (7.2)), but in the stochastic case there is dependence, see (7.9).

To illustrate the above results, consider the quadratic penalty (2.35). Then the BCE functional (for the RV D) is

$$b(y) = dy + \frac{1}{2}\sigma^2 y^2$$

(for the quadratic penalty, the approximation (3.9) is exact). The equation (7.9) determining the optimal \bar{S} is here

$$\bar{S}^2 h \theta - 2\sigma^2 \frac{k^2}{\theta \bar{S}} = 2k(d + \sigma^2 c) \quad (7.10)$$

where $\theta = 1 + \frac{h}{p}$. From (7.10) it follows that

\bar{S} increases with the expected demand d ,

\bar{S} increases with the demand variance σ^2 .

Since $\bar{Q} = \bar{S} - \bar{s} = \bar{S}\theta$ the same conclusions hold for the effects on the optimal order quantity \bar{Q} . The result that \bar{Q} increases with σ^2 is intuitively clear: The risk-averse decision maker keeps a larger inventory to cope with increased

demand fluctuations.

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APPENDIX A. PROOF OF THEOREM 2.

(a) By (3.1) and (1.9), $s(\cdot)$ is the pointwise supremum of concave functionals, hence concave. The rest of (a) is proved as in Lemma 2.3(a).

(b) For $y = 0$, (3.4) gives

$$EU'(-z_S(0)) = 1$$

or $U'(-z_S(0)) = 1$, proving that $z_S(0) = 0$. From (3.3) it follows then that $s(0) = 0$.

Differentiating (3.4) with respect to y gives

$$EU''(y \cdot Z - z_S(y)) (Z - \nabla z_S(y)) = 0$$

which at $y = 0$ becomes

$$EU''(0) (EZ - \nabla z_S(0)) = 0$$

proving that $\nabla z_S(0) = \mu$. Then, by differentiating (3.3) at $y = 0$ we get $\nabla s(0) = 0$.

The expressions for $\nabla^2 z_S(0)$ and $\nabla^2 s(0)$ follow similarly by differentiating (3.4) and (3.3) twice at $y = 0$.

(c) and (d) are proved analogously. □

APPENDIX B. PROPERTIES OF S_{U_α} .

LEMMA 2. Let $U \in \mathcal{U}$, $\alpha > 0$, and let U_α be defined by (2.17). Then:

(a) $S_{U_\alpha}(Z)$ is monotone decreasing in α .

(b) $\lim_{\alpha \rightarrow 0} S_{U_\alpha}(Z) = \mu$.

(c) If U is **essentially smooth** ([13], p. 251),

$$\lim_{\alpha \rightarrow \infty} S_{U_\alpha}(Z) = z_{\min}.$$

PROOF. (a) We prove that U_α is monotone decreasing in α . Indeed,

$$\begin{aligned} \frac{d}{d\alpha} U(\alpha z) &= \frac{1}{\alpha^2} [\alpha z U'(\alpha z) - U(\alpha z)] \\ &\leq \frac{1}{\alpha^2} [-U(0)] \\ &= 0 \quad \text{since } U(0) = 0. \end{aligned}$$

The above inequality is the gradient inequality for the concave function U . The monotonicity of U_α , as a function of α , is inherited by S_{U_α} of (2.18).

(b) By L'Hopital's rule, since $U'(0) = 1$,

$$\lim_{\alpha \rightarrow 0} U_\alpha(z) = z, \quad \forall z$$

from which (b) follows.

(c) Since U is essentially smooth it can be shown that

$$\lim_{\alpha \rightarrow \infty} U_\alpha(z) = -\infty, \quad \forall z < 0$$

from which (c) follows. □

APPENDIX C. RESULTS FROM §6.

PROOF OF PROPOSITION 7. (a) The objective function in (6.3)

$$h(k) := (1-k)W_0\tau + s(W_0k)$$

is concave, by Theorem 2(a). Hence, the optimal solution x^* is an inner solution, i.e. $0 < k^* < 1$ if and only if

$$h'(0) > 0 \text{ and } h'(1) < 0 \quad (\text{C.1})$$

Now

$$h(k) = -W_0\tau + W_0s'(W_0k) \quad (\text{C.2})$$

which becomes, upon substitution of the computed expression for $s'(\cdot)$,

$$h(k) = -W_0\tau + W_0EtU'(W_0kt - \eta(W_0k)) \quad (\text{C.3})$$

where $\eta(q)$ is the unique solution of

$$EU'(qt - \eta) = 1 \quad (\text{C.4})$$

Therefore

$$h'(0) = -W_0\tau + W_0E(t)$$

$$h'(1) = -W_0\tau + W_0EtU'(W_0 - \eta(W_0))$$

and (C.1) is equivalent to (6.4).

(b) Let $k(\tau)$ be the optimal solution of (6.3) for given τ , i.e. $h'(k(\tau)) = 0$, or using (C.3),

$$-\tau + E\{tU'(W_0k(\tau)t - \eta(W_0k(\tau)))\} \equiv 0$$

Differentiating this identity (in τ) with respect to τ , we obtain

$$-1 + E\{tW_0(k'(\tau)t - k'(\tau)\eta'(W_0k(\tau)))U''\} = 0$$

or

$$k'(\tau)W_0Et(t - \eta')U'' = 1 \quad (\text{C.5})$$

Now, the second order condition for the maximality of $k(\tau)$ is

$$0 > h''(k) = W_0E\{tW_0(t - \eta')U''\} \quad (\text{C.6})$$

Therefore, $k'(\tau)$ is multiplied in (C.5) by a negative number, and consequently

$$k'(\tau) < 0$$

proving that $k(\tau)$ [1 - $k(\tau)$], the proportion invested in the risky [safe] asset, is a decreasing [increasing] function of τ , the safe asset return. \square

PROOF OF PROPOSITION 8. Let $k = k(W_0)$ be the optimal solution of (6.3), i.e. $h'(k(W_0)) = 0$, or using (C.3)

$$-\tau + E\{tU'(W_0k(W_0)t - \eta(W_0k(W_0)))\} \equiv 0 \quad (\text{C.7})$$

Differentiating this identity (in W_0) we get

$$Et[k(W_0) + W_0k'(W_0)][t - \eta'(W_0k(W_0))]U'' = 0$$

or

$$k'W_0Et(t - \eta')U'' = - EtkU'' \quad (C.8)$$

By the second order optimality condition (C.6) it follows that, in (C.8), k' is multiplied by a negative number. Since the right hand side of (C.8) is positive ($t, k > 0, U'' < 0$), it follows that

$$k'(W_0) < 0$$

□

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